Exercise 50

Use logarithmic differentiation to find the derivative of the function.

$$y = (\ln x)^{\cos x}$$

Solution

Take the natural logarithm of both sides and use the properties of logarithms to simplify the right side.

$$\begin{aligned}
\ln y &= \ln(\ln x)^{\cos x} \\
&= (\cos x) \ln \ln x
\end{aligned}$$

Differentiate both sides with respect to x.

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(\cos x \ln \ln x) \\
\frac{1}{y} \cdot \frac{d}{dx}(y) &= \left[\frac{d}{dx}(\cos x)\right] \ln \ln x + (\cos x)\left[\frac{d}{dx}(\ln \ln x)\right] \\
\frac{1}{y} \cdot \frac{dy}{dx} &= (-\sin x) \ln \ln x + (\cos x)\left[\frac{1}{\ln x} \cdot \frac{d}{dx}(\ln x)\right] \\
\frac{1}{y}\frac{dy}{dx} &= -\sin x \ln \ln x + (\cos x)\left[\frac{1}{\ln x} \cdot \left(\frac{1}{x}\right)\right] \\
\frac{dy}{dx} &= y\left(-\sin x \ln \ln x + \frac{\cos x}{x \ln x}\right) \\
&= (\ln x)^{\cos x}\left(-\sin x \ln \ln x + \frac{\cos x}{x \ln x}\right) \\
&= (\ln x)^{\cos x}\left(-\frac{x \sin x \ln x \ln \ln x}{x \ln x} + \frac{\cos x}{x \ln x}\right) \\
&= (\ln x)^{\cos x}\left(\frac{\cos x - x \sin x \ln x \ln \ln x}{x \ln x}\right) \\
&= \frac{\cos x - x \sin x \ln x \ln \ln x}{x(\ln x)^{1-\cos x}}
\end{aligned}$$